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Tensor product and permutation branes on the torus

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Abstract: We consider B-type D-branes in the Gepner model consisting of two minimal models at $k = 2$. This Gepner model is mirror to a torus theory. We establish the dictionary identifying the B-type D-branes of the Gepner model with A-type Neumann and Dirichlet branes on the torus.

Keywords: D-branes, Topological Strings, Conformal Field Models in String Theory.

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Contents

1. Introduction

D-branes in models with $N = (2, 2)$ world-sheet supersymmetry have been studied in various approaches and at different points in moduli space, and it has been fruitful to combine several viewpoints (see e.g. [1−7]). In this paper we will study an example of the relationship between D-branes in Gepner models (for some early work see [8]), those of the corresponding geometric compactification (see e.g. [9, 10]), and matrix factorisations of the equivalent Landau-Ginzburg theory that were first studied in [11−13].

There are two classes of branes that preserve half of the $N = 2$ supersymmetry [14]; these are called A-type and B-type, and are related by mirror symmetry. In the following we are going to consider the B-type branes of a Gepner model involving two minimal models at $k = 2$, giving a total central charge of $c = 3$. These branes have an interpretation in terms of A-type branes in the corresponding mirror, which is a torus theory [15].

We shall construct an explicit map between certain branes in the Gepner model (tensor product and permutation branes [16, 17]) and those of the torus, matching the minimal model labels of the former with angles, positions, and Wilson lines of the latter. This will be done by writing the boundary states in either theory in terms of the Ishibashi states of the diagonal $N = 2$ theory at $c = 3$. The Gepner model, on the other hand, can be described topologically by an orbifold of the Landau-Ginzburg theory with superpotential $W = x_1^4 + x_2^4 + z^2$ (see e.g. [15, 18]), where the branes are described by matrix factorisations of the superpotential. In a second step, we shall identify the Gepner branes with matrix factorisations of the Landau-Ginzburg theory [19, 20].

A similar analysis has already been performed in [21]. There the dictionary between the tensor product Gepner branes and the torus branes was studied by comparing the self-overlap of the boundary states. This leads to an identification of the angle of the Gepner branes in the torus description. Here we shall be more explicit; in particular we shall also determine the relative positions and Wilson lines of the Gepner branes, and we shall also discuss permutation branes.

The relation between the tensor product branes of the Gepner model and the branes of the LG theory with superpotential $W = x_1^4 + x_2^4 + z^2$ was also studied in [22]. Finally, the branes of the LG theory with superpotential $W = x_1^4 + x_2^4$, which is mirror to the $\mathbb{Z}_4$ orbifold of the torus, were related in [23] by matching intersection matrices and the coupling to RR-primary fields.

The organisation of the paper is as follows: In section 2, we set up our notation for the torus theory and its A-type D-branes; we also consider two $\mathbb{Z}_4$ symmetries whose action on the branes has a geometric interpretation. In section 3 the same is done for the corresponding Gepner model and its B-type tensor product and permutation branes. In particular we give the explicit formulae and propose two $\mathbb{Z}_4$ symmetries, one being the quantum symmetry of the orbifold, that correspond to those of the torus. Section 4 explains the matching of the branes on both sides, and in section 5 we relate the Gepner branes to matrix factorisations of the corresponding superpotential. Section 6 contains some conclusions.

2. The torus $\mathcal{T}^2$

Let us begin by setting up our conventions for the conformal field theory on the torus $\mathcal{T}^2$. The torus shall be rectangular, and have both radii at the self-dual value. This theory is the mirror of the Gepner model considered in section 3.

2.1 Space of states

The torus is given by two free bosonic fields $X^1(z, \bar{z})$, $X^2(z, \bar{z})$, and two fermionic fields $\psi^1(z) + \tilde{\psi}^1(\bar{z})$, $\psi^2(z) + \tilde{\psi}^2(\bar{z})$, where we have explicitly written out the chiral and antichiral parts. The bosonic fields are each compactified on a circle of self-dual radius $R = 1$ (for $\alpha' = 1$):

$$X^i(z, \bar{z}) \sim X^i(z, \bar{z}) + 2\pi \qquad (i = 1, 2)\,.$$

We can complexify these fields as

$$X^\pm := \frac{1}{\sqrt{2}}(X^1 \pm iX^2)\,, \quad \psi^\pm := \frac{1}{\sqrt{2}}(\psi^1 \pm i\psi^2)\,, \quad \tilde{\psi}^\pm := \frac{1}{\sqrt{2}}(\tilde{\psi}^1 \pm i\tilde{\psi}^2)\,.$$

For the bosonic fields, the derivatives with respect to z ($\bar{z}$) are chiral (antichiral) fields with mode expansion

$$\partial_z X^{\pm} = -i \sum_{n\in\mathbb{Z}} \alpha_n^{\pm}\, z^{-n-1}\,, \quad \partial_{\bar{z}} X^{\pm} = -i \sum_{n\in\mathbb{Z}} \tilde{\alpha}_n^{\pm}\, \bar{z}^{-n-1}\,; \tag{2.1}$$

the chiral fermionic fields have the expansion

$$\psi^{\pm} = \sum_r \psi_r^{\pm}\, z^{-r-\frac{1}{2}}\,, \quad \tilde{\psi}^{\pm} = \sum_r \tilde{\psi}_r^{\pm}\, \bar{z}^{-r-\frac{1}{2}}\,, \tag{2.2}$$

where $r \in \mathbb{Z}$ in the Ramond sector and $r \in \mathbb{Z} + \frac{1}{2}$ in the Neveu-Schwarz sector. Due to the compactification, the ground states in the space of states have momenta given by momentum and winding quantum numbers p_i and w_i,

$$P_i^L = \frac{1}{\sqrt{2}}(p_i + w_i), \quad P_i^R = \frac{1}{\sqrt{2}}(p_i - w_i) \quad (i = 1,\, 2)\,. \tag{2.3}$$

The superscripts L and R of the center of mass momenta P refer to left- and right-moving fields.

We will be interested in the $N = 2$ supersymmetry of this theory. The Verma module with respect to the $N = 2$ generators on each of the ground states (2.3), except for the vacuum state $p_i = w_i = 0$, forms an irreducible $N = 2$ highest weight representation at $c = 3$. In the NS sector, the corresponding highest weight state has conformal dimension H ($\tilde{H}$) and $\mathrm{U}(1)$ charge Q ($\tilde{Q}$) for the left-(right-)movers, with

$$H = \frac{1}{4}((p_1 + w_1)^2 + (p_2 + w_2)^2)\,, \quad Q = 0\,, \tag{2.4}$$

$$\tilde{H} = \frac{1}{4}((p_1 - w_1)^2 + (p_2 - w_2)^2)\,, \quad \tilde{Q} = 0\,. \tag{2.5}$$

Highest weights and charges of the R sector states are reached by spectral flow, which gives rise to a representation at highest weight $H+\frac{1}{8}$ for every NS representation at highest weight $H > 0$. We will use the convention that we label a Ramond representation by conformal dimension and charge of the highest weight vector which is annihilated by the mode G_0^+.

Ground states with momenta as in (2.3) will be denoted

$$|p_1,\, w_1,\, p_2,\, w_2\rangle_{\mathrm{NS,\,R}} \tag{2.6}$$

for momentum quantum numbers p_i and winding numbers w_i. We will drop the R or NS index when unnecessary.

The Verma modules built on the vacuum states $p_i = w_i = 0$ are reducible in both the NS and the R sector, as in the uncompactified case [24]. In the NS sector, highest weights and charges of these representations are given by

$$H = 0\,, \quad Q = 0\,; H = \frac{2|n| - 1}{2}\,, \quad Q = \mathrm{sign}(n) \quad \text{for } n \in \mathbb{Z} \setminus \{0\}\,. \tag{2.7}$$

The R sector representations follow again with the help of the spectral flow.

These representations are generated in the NS sector from the singular vectors of the $N = 2$ vacuum Verma module

$$\begin{aligned} &(\alpha^+_{-1})^{n-1}\psi^+_{-\frac{1}{2}}|0,\,0,\,0,\,0\rangle_{\rm NS}\,,\\ &(\alpha^-_{-1})^{n-1}\psi^-_{-\frac{1}{2}}|0,\,0,\,0,\,0\rangle_{\rm NS}\,, \end{aligned} \tag{2.8}$$

for $n \in \mathbb{N}$; we will use the short-hand notation

$$|n\rangle_{\rm NS} = \begin{cases} |\frac{2|n|-1}{2},\,1\rangle_{\rm NS} & \text{for } n > 0\,,\\ |0,\,0\rangle_{\rm NS} & \text{for } n = 0\,,\\ |\frac{2|n|-1}{2},\,-1\rangle_{\rm NS} & \text{for } n < 0\,, \end{cases} \tag{2.9}$$

where the right-hand side gives the conformal dimension and charge of the corresponding highest weight vector. Here, the states with $n > 0$ denote the states of the first line in (2.8), and those with $n < 0$ the states of the second line in (2.8).

In the R sector, the singular vectors are

$$\begin{aligned} &(\alpha^+_{-1})^{n-1}\psi^+_{-1}|0,\,0,\,0,\,0\rangle_{\rm R}\,, \qquad &&(\alpha^+_{-1})^{n}|0,\,0,\,0,\,0\rangle_{\rm R}\,;\\ &(\alpha^-_{-1})^{n-1}\psi^-_{-1}\psi^-_0|0,\,0,\,0,\,0\rangle_{\rm R}\,, \qquad &&(\alpha^-_{-1})^{n}\psi^-_0|0,\,0,\,0,\,0\rangle_{\rm R}\,, \end{aligned} \tag{2.10}$$

where $|0,\,0,\,0,\,0\rangle_{\rm R}$ is the free field ground state $|\frac{1}{8},\,\frac{1}{2}\rangle_{\rm R}$. We will use a short-hand notation analogous to the NS case, namely

$$\begin{aligned} |n,+\rangle_{\rm R} &= \begin{cases} |\frac{1}{8},\,\frac{1}{2}\rangle_{\rm R} & (n=0)\\ |n+\frac{1}{8},\,\frac{3}{2}\rangle_{\rm R} & (n\in\mathbb{N}) \end{cases}\,,\\ |n,-\rangle_{\rm R} &= \begin{cases} |\frac{1}{8},\,-\frac{1}{2}\rangle_{\rm R} & (n=0)\\ |n+\frac{1}{8},\,-\frac{3}{2}\rangle_{\rm R} & (n\in\mathbb{N}) \end{cases}\,. \end{aligned} \tag{2.11}$$

2.2 Two $\mathbb{Z}_4$ symmetries on $\mathcal{T}^2$

We note two $\mathbb{Z}_4$ symmetries that we will identify in section 4 with symmetries of the corresponding Gepner model.

The rotation group $\mathbb{Z}_4$ acts naturally on the bosonic torus fields when the action of its generator g on the fields is given by

$$\begin{aligned} g(X^1) &= -X^2\,, \qquad & g(X^2) &= X^1\,,\\ g(\psi^1) &= -\psi^2\,, \qquad & g(\psi^2) &= \psi^1\,, \end{aligned}$$

which in terms of the complexified fields reads

$$g(X^\pm) = e^{\pm i\frac{\pi}{2}}X^\pm\,, \qquad g(\psi^\pm) = e^{\pm i\frac{\pi}{2}}\psi^\pm\,.$$

A little care is required when we define the phase of the action of g on the ground states with non-vanishing momentum (2.6). In the NS sector, we can define

$$g|p_1,\,w_1,\,p_2,\,w_2\rangle_{\rm NS} = |-p_2,\,-w_2,\,p_1,\,w_1\rangle_{\rm NS}\,. \tag{2.12}$$

In the R sector, where the ground states form a tensor product of two two-dimensional representations of the Dirac algebra, we must include an appropriate phase.

The highest weight states in the vacuum sectors obtain a phase under the $\mathbb{Z}_4$ action according to (2.8), (2.10):

$$
\begin{aligned}
g|n\rangle_{\mathrm{NS}} &= e^{i\pi n}|n\rangle_{\mathrm{NS}} && (n \in \mathbb{Z})\,, \\
g|n,\,\pm\rangle_{\mathrm{R}} &= e^{\pm i\pi(n+\frac{1}{2})}|n,\,\pm\rangle_{\mathrm{R}} && (n \in \mathbb{N}_0)\,.
\end{aligned}
\tag{2.13}
$$

A linear combination of ground states which is an eigenstate of eigenvalue $e^{i\frac{\pi}{2}t}$ for $0 \le t \le 3$ with respect to this $\mathbb{Z}_4$ symmetry will be denoted with a superscript t:

$$
|p_1,\,w_1,\,p_2,\,w_2\rangle^t = \frac{1}{2}\sum_{n=0}^{3} e^{-i\frac{\pi}{2}tn} g^n |p_1,\,w_1,\,p_2,\,w_2\rangle\,. \tag{2.14}
$$

The other symmetry is a $\mathbb{Z}_4$ symmetry involving T-duality, which we will call $\mathbb{Z}_4'$ in order to distinguish it from the previous one. Denoting its generator g', it acts on the ground states (2.6) as

$$
g'|p_1,\,w_1,\,p_2,\,w_2\rangle_{\mathrm{NS}} = (-1)^{p_1+p_2}|w_1,\,p_1,\,w_2,\,p_2\rangle_{\mathrm{NS}}\,. \tag{2.15}
$$

In the vacuum sectors, this symmetry has the same effect as (2.13).

An eigenstate with respect to both symmetries is denoted

$$
|p_1,\,w_1,\,p_2,\,w_2\rangle^{t,m} = \frac{1}{4}\sum_{n=0}^{3}\sum_{s=0}^{3} e^{-i\frac{\pi}{2}(st+mn)} g^s (g')^n |p_1,\,w_1,\,p_2,\,w_2\rangle\,. \tag{2.16}
$$

The superscript on the left-hand side indicates the eigenvalues $e^{i\frac{\pi}{2}t}$ under g and $e^{i\frac{\pi}{2}m}$ under g'.

The first $\mathbb{Z}_4$ action can be interpreted geometrically as a rotation by 90 degrees. The action of $\mathbb{Z}_4'$ amounts to a T-duality transformation in both directions, and the phase can be seen as a shift $X^i(z,\bar{z}) \mapsto X^i(z,\bar{z}) + \pi$ in both directions, i.e. as $X^i_{\mathrm{L,R}} \mapsto X^i_{\mathrm{L,R}} + \frac{\pi}{2}$.

2.3 The $N=2$ boundary states on $\mathcal{T}^2$

We are interested in boundary states on the torus that satisfy the $N=2$ boundary conditions of type A,

$$
\begin{aligned}
(L_n - \tilde{L}_{-n})\,\|A\rangle\rangle &= 0\,, \\
(J_n - \tilde{J}_{-n})\,\|A\rangle\rangle &= 0\,, \\
(G^{\pm}_r + i\eta \tilde{G}^{\mp}_{-r})\,\|A\rangle\rangle &= 0\,,
\end{aligned}
\tag{2.17}
$$

with spin structure $\eta \in \{\pm 1\}$. The zero mode condition is $H = \tilde{H}$ and $Q = \tilde{Q}$, which means

$$
p_1 w_1 = -p_2 w_2 \tag{2.18}
$$

in terms of the ground state quantum numbers. For non-vanishing momenta, there is — up to a phase — a unique Ishibashi state in these representations, which we will denote by [25, 26]

$$|p_1,\, w_1,\, p_2,\, w_2;\, \eta\rangle\rangle_{\mathrm{NS,\,R}}\,. \tag{2.19}$$

The subscript is to be understood as specifying either the NS-NS or the R-R sector. We can fix the relative normalisations between Ishibashi states at given highest weight by demanding that these states transform under the $\mathbb{Z}_4$ symmetries in the same way (2.12), (2.15) as the NS ground states, i.e. by setting

$$\begin{aligned} g|p_1,\, w_1,\, p_2,\, w_2\rangle\rangle_{\mathrm{NS,\,R}} &= |-p_2,\, -w_2,\, p_1,\, w_1\rangle\rangle_{\mathrm{NS,\,R}}\,, \\ g'|p_1,\, w_1,\, p_2,\, w_2\rangle\rangle_{\mathrm{NS,\,R}} &= (-1)^{p_1+p_2}|w_1,\, p_1,\, w_2,\, p_2\rangle\rangle_{\mathrm{NS,\,R}}\,. \end{aligned} \tag{2.20}$$

In the vacuum sector, the representations containing the same singular vectors in the left- and the right-moving part of the theory are isomorphic, so that we have an Ishibashi state for every left-moving irreducible highest weight representation. We will denote the Ishibashi states in the vacuum sectors analogously to the left-moving ground states by

$$|n;\, \eta\rangle\rangle_{\mathrm{NS}}, \qquad |n,\, \pm;\, \eta\rangle\rangle_{\mathrm{R}}, \tag{2.21}$$

where $n \in \mathbb{Z}$ in the NS-NS and $n \in \mathbb{N}_0$ in the R-R sector.

It was shown in [24] that the $N = 2$ boundary states on the torus can all be expressed in terms of the usual Neumann branes with electric fields. The Neumann gluing conditions with flux ϕ,

$$\begin{aligned} (\alpha_n^{\pm} + e^{\mp i\phi}\, \tilde{\alpha}_{-n}^{\mp})\, \|A\rangle\rangle = 0\,, \\ (\psi_r^{\pm} + i\eta e^{\mp i\phi}\, \tilde{\psi}_{-r}^{\mp})\, \|A\rangle\rangle = 0\,, \end{aligned} \tag{2.22}$$

imply the $N = 2$ gluing conditions (2.17) for every (real) value of ϕ. On the other hand, every fundamental $N = 2$ boundary state on the torus is actually a state satisfying (2.22) for a specific flux ϕ. In fact, for non-vanishing ground-state momentum, any $N = 2$ Ishibashi state $|p_1,\, w_1,\, p_2,\, w_2;\, \eta\rangle\rangle$ actually defines a $\mathrm{U}(1)$ Ishibashi state satisfying (2.22), with ϕ given by

$$\tan\left(\frac{\phi}{2}\right) = \frac{p_1}{p_2}\,. \tag{2.23}$$

This expression allows us to interpret the quantity ϕ as an angle in the space of momentum quantum numbers. In the NS sector, a convenient notation is to label the $\mathrm{U}(1)$ states by two coprime momentum quantum numbers $\hat{p}_1 \in \mathbb{Z}$, $\hat{p}_2 \in \mathbb{N}_0$, and two integers a, b,

$$|a,\, b,\, \phi;\, \eta\rangle\rangle_{\mathrm{NS}} := |a\hat{p}_1,\, b\hat{p}_2,\, a\hat{p}_2,\, -b\hat{p}_1;\, \eta\rangle\rangle_{\mathrm{NS}}\,, \tag{2.24}$$

where the right-hand side is in the notation (2.19), and ϕ satisfies (2.23), i.e. $\tan(\phi/2) = \hat{p}_1/\hat{p}_2$.

In the R sector, we have to be more careful. It turns out that we can avoid the action of g to look quite tedious in both sectors by extending the angle ϕ to take values in the interval $(-2\pi, 2\pi]$, i.e. to enlarge its period to 4π. We will hence associate to every Ishibashi state in the representation labelled by $p_1 = a\hat{p}_1$, $w_1 = b\hat{p}_2$, $p_2 = a\hat{p}_2$, $w_2 = -b\hat{p}_1$ the two notations

$$|a,\, b,\, \phi\rangle\rangle_{\rm R} = -|-a,\, -b,\, \phi + 2\pi\rangle\rangle_{\rm R}\,, \tag{2.25}$$

and do the same in the NS sector, but without the relative minus sign. Arranging the signs, the action of g and g' on the Ishibashi states in the new notation can be written as

$$\begin{aligned} g|a,\, b,\, \phi\rangle\rangle &= |a,\, b,\, \phi + \pi\rangle\rangle\,, \\ g'|a,\, b,\, \phi\rangle\rangle &= |-b,\, a,\, \phi + \pi\rangle\rangle \end{aligned}$$

in both the R and the NS sector.

In the vacuum sectors, the $N = 2$ Ishibashi states transform analogously to their respective ground states, i.e.

$$\begin{aligned} g|n;\, \eta\rangle\rangle_{\rm NS} &= e^{i\pi n}|n;\, \eta\rangle\rangle_{\rm NS} && (n \in \mathbb{Z}) \\ g|n,\, \pm;\, \eta\rangle\rangle_{\rm R} &= e^{\pm i\pi(n+\frac{1}{2})}|n,\, \pm;\, \eta\rangle\rangle_{\rm R} && (n \in \mathbb{N}_0)\,. \end{aligned} \tag{2.26}$$

We will now fix the remaining phases in the definition of our Ishibashi states. In the vacuum sectors, this is done by setting

$$\begin{aligned} \|\text{Neumann};\, \eta\rangle\rangle_{\rm NS} &= \mathcal{N}\sum_{n\in\mathbb{Z}} |n;\, \eta\rangle\rangle_{\rm NS} \qquad \text{and} \\ \|\text{Neumann};\, \eta\rangle\rangle_{\rm R} &= \mathcal{N}\sum_{n\in\mathbb{N}_0} \Big(|n,\, +;\, \eta\rangle\rangle_{\rm R} + |n,\, -;\, \eta\rangle\rangle_{\rm R}\Big)\,, \end{aligned}$$

where $\|\text{Neumann};\, \eta\rangle\rangle_{\rm NS,\, R}$ is the free field Neumann vacuum boundary state. The phases of the other Ishibashi states are determined by writing the boundary states in the following form [26]:

$$\begin{aligned} \|A,\, B,\, \phi, \epsilon;\, \eta\rangle\rangle = \mathcal{N}(\phi)\bigg\{ &\sum_{n\in\mathbb{Z}} e^{in\phi}\, |n;\, \eta\rangle\rangle_{\rm NS} + \sum\nolimits_{(a,b)\in\mathbb{Z}^2\setminus\{(0,0)\}} e^{iAa+iBb}|a,\, b,\, \phi;\, \eta\rangle\rangle_{\rm NS} \\ &+ i\epsilon\bigg[\sum_{n\in\mathbb{N}_0} \Big(e^{i(n+\frac{1}{2})\phi}\, |n,\, +;\, \eta\rangle\rangle_{\rm R} + e^{-i(n+\frac{1}{2})\phi}\, |n,\, -;\, \eta\rangle\rangle_{\rm R}\Big) \\ &+ \sum\nolimits_{(a,b)\in\mathbb{Z}^2\setminus\{(0,0)\}} e^{iAa+iBb}|a,\, b,\, \phi;\, \eta\rangle\rangle_{\rm R}\bigg]\bigg\}\,. \end{aligned} \tag{2.27}$$

In this notation, A is the relative position of the brane, B its Wilson line, and $\epsilon \in \{\pm 1\}$ distinguishes a brane from its respective anti-brane. The equivalences in this notation are

$$\begin{aligned} &\|A + 2\pi,\, B,\, \phi,\, \epsilon;\, \eta\rangle\rangle = \|A,\, B + 2\pi,\, \phi,\, \epsilon;\, \eta\rangle\rangle = \|A,\, B,\, \phi,\, \epsilon;\, \eta\rangle\rangle\,, \\ &\|A,\, B,\, \phi + 2\pi,\, \epsilon;\, \eta\rangle\rangle = \|A,\, B,\, \phi,\, -\epsilon;\, \eta\rangle\rangle\,. \end{aligned}$$

The $\mathbb{Z}_4$ symmetries from above act as

$$\begin{aligned} g\|A,\, B,\, \phi,\, \epsilon;\, \eta\rangle\rangle &= \|A,\, B,\, \phi + \pi,\, \epsilon;\, \eta\rangle\rangle\,, \\ g'\|A,\, B,\, \phi,\, \epsilon;\, \eta\rangle\rangle &= \|-B,\, A,\, \phi + \pi,\, \epsilon;\, \eta\rangle\rangle\,. \end{aligned} \tag{2.28}$$

After these preperations, we now turn to the Gepner description of the mirror theory [15, 27].

3. $\mathcal{T}^2$ as a Gepner model

The Gepner construction consists of a free conformal field theory describing an uncompactified D-dimensional space-time, with an interior conformal field theory built by means of a tensor product of $N = 2$ minimal models [28]. An orbifold ensures that the complete theory is a consistent superstring theory with space-time supersymmetry and a modular invariant partition function [29].

In the following we shall only consider the internal part of the theory, namely a tensor product of two minimal models at level $k = 2$, which together give a central charge $c = 3$. In order to relate this theory to the torus, we need to perform an orbifold that can be understood in terms of a simple current extension [30 − 32].

For the tensor product of two minimal models at level $k = 2$, there are two primary fields that generate the simple current subgroup. In the coset notation, these fields are[1]

$$u = (\Phi_2^{0,0},\, \Phi_2^{0,2})\,, \qquad w = (\Phi_0^{0,2},\, \Phi_0^{0,2})\,, \tag{3.1}$$

again for general even k. Projection onto zero monodromy charge with respect to the current u amounts to keeping only fields with

$$m_1 + m_2 = \frac{k+2}{2} s_1 \mod k+2\,, \tag{3.2}$$

and the charge projection with respect to the simple current w provides the exclusion of NS-R coupling. The simple current extension therefore leaves us with the space of states

$$\bigoplus_{\substack{[l_1,m_1,s_1],\\ [l_2,m_2,s_2],\\ t\in\mathbb{Z}_{k+2},\\ \tilde{s}_i = s_i \bmod 2}} (l_1,m_1,s_1)\otimes(l_2,m_2,s_2)\otimes(l_1,m_1-2t,\tilde{s}_1)\otimes(l_2,m_2-2t,\tilde{s}_2)\,, \tag{3.3}$$

where the sum runs over equivalence classes (denoted by the square brackets) of minimal model representations with coset labels (l,m,s), subject to fermion alignment $s_1 - s_2 = 0$ mod 2 to prohibit the NS-R coupling, and to charge projection (3.2) for zero monodromy charge. The first two factors in (3.3) refer to left-moving and the second two to right-moving representations.

For $k = 2$, the diagonal algebra of a tensor product of two minimal models is an $N = 2$ algebra at $c = 3$, and (3.3) decomposes into a direct sum of representations of the diagonal algebra. The diagonal representations corresponding to highest weight vectors of lowest conformal dimension with respect to the construction (3.3) at $k = 2$ can be read off from the low-level expansion of the characters. One finds

$$\begin{aligned}
\left(\frac{1}{4},\frac{1}{2}\right)\otimes\left(\frac{1}{4},\frac{1}{2}\right) &= \left(\frac{1}{2},1\right)\oplus(1,\,0)\oplus(2,\,0)\oplus\dots\,,\\
\left(\frac{1}{2},0\right)\otimes(0,\,0) &= \left(\frac{1}{2},0\right)\oplus\left(\frac{5}{2},0\right)\oplus\left(\frac{5}{2},0\right)\oplus\dots\,,\\
\left(\frac{1}{8},\frac{1}{4}\right)\otimes\left(\frac{1}{8}.-\frac{1}{4}\right) &= \left(\frac{1}{4},0\right)\oplus\left(\frac{5}{4},0\right)\oplus\left(\frac{9}{4},0\right)\oplus\left(\frac{13}{4},0\right)\oplus\dots\,,
\end{aligned} \tag{3.4}$$

[1] see the appendix for our conventions on the coset labels.

where the left-hand side gives the highest weights and charges of the minimal model representations, and the right-hand side those of the representations of the diagonal algebra. The other tensor products of minimal model representations that appear in the theory are linked to those in (3.4) by spectral flow, where the same flow parameter is applied on both factors on the left hand side, as well as on the representations appearing in the sum on the right-hand side. From the expansion one can also guess a general formula for the decomposition (3.4) (compare with the case considered in [33]); this is described in the appendix. However (3.4) already contains all the information we are going to need in the following.

In the tensor product of minimal models at $k = 2$, the primary fields with $l = \frac{k}{2} = 1$ are fixed points under the action of u^2. Since u generates a cyclic $\mathbb{Z}_4$ subgroup of the simple current group, the stabiliser of these fields is isomorphic to $\mathbb{Z}_2$. Due to this fixed point we can not directly apply the formulae for the tensor product branes from [19], but will have to resolve the S-matrix [31, 32]. The formulae for the branes will be given in the following subsections (where we will construct the tensor product states in a similar way as in [34]).

Before we come to them, let us point out the two $\mathbb{Z}_{k+2}$ symmetries in the theory (3.3) that we will use later on to fix the map between Ishibashi states of the Gepner model and Ishibashi states on the torus. The first is the quantum symmetry, i.e. the symmetry that is used to undo the orbifold we have just achieved by the simple current extension. It divides the space of states into the twist sectors $t = 0, \ldots, k+1$ by acting as a phase $e^{\frac{2\pi i}{k+2}t}$ on the states within the respective sector. The second symmetry acts as a phase $e^{\frac{2\pi i}{k+2}m_1}$ on a state with left-moving labels $(l_1, m_1, s_1) \otimes (l_2, m_2, s_2)$. By considering states in representations of low-lying ground-state momenta, one can see that these two symmetries are in fact just the symmetries $\mathbb{Z}_4$ and $\mathbb{Z}_4'$ from section 2.2.

Incidentally, the requirement that the quantum $\mathbb{Z}_4$ symmetry acts geometrically on the torus side requires that we make use of mirror symmetry and relate the B type branes of the Gepner model to A type branes on the torus.

3.1 Boundary states on the Gepner model

Supersymmetric B-type boundary states in the theory (3.3) satisfy the gluing conditions

$$\begin{aligned} \left(L_n^{(1)} + L_n^{(2)} - \tilde{L}_{-n}^{(1)} - \tilde{L}_{-n}^{(2)}\right) \|B\rangle\rangle &= 0\,, \\ \left(J_n^{(1)} + J_n^{(2)} + \tilde{J}_{-n}^{(1)} + \tilde{J}_{-n}^{(2)}\right) \|B\rangle\rangle &= 0\,, \\ \left(G_r^{\pm\,(1)} + G_r^{\pm\,(2)} + i\eta(\tilde{G}_{-r}^{\pm\,(1)} + \tilde{G}_{-r}^{\pm\,(2)})\right) \|B\rangle\rangle &= 0\,. \end{aligned} \tag{3.5}$$

Among these states we will focus on the tensor product and the permutation branes [19], and give a map between them and certain boundary states on the torus.

3.1.1 Tensor product boundary states

Tensor product branes satisfy (3.5) separately for the two tensor product factors (1) and (2). In the sector of the form

$$(l_i, m_i, s_i) \otimes (l_i, m_i - 2t, \tilde{s}_i) \quad (i = 1,\, 2) \tag{3.6}$$

we find a B-type Ishibashi state if there exists an automorphism between the left- and the right-moving representation which takes the coset labels $(l,\, m,\, s)$ to

$$\tilde{l} = l\,, \quad \tilde{m} = -m\,, \quad \tilde{s} = -s \tag{3.7}$$

up to field identification. In other words, we can construct a B-type Ishibashi state when a left-moving representation $(l,\, m,\, s)$ is tensored to a right-moving representation $(\tilde{l},\, \tilde{m},\, \tilde{s})$ in (3.6), which amounts to demanding that

$$m_i = t \mod k+2\,, \qquad s_i = -\tilde{s}_i\,. \tag{3.8}$$

There is a subtlety when $l_i = \frac{k}{2}$, where the labels of the right-moving representation in (3.6) may encode the conjugate representation, but (3.7) is only met after a field identification. Note that our convention to use the same l labels in the left- and the right-moving representation prevents us from overlooking this possibility in the other cases. There exist therefore additional Ishibashi states for

$$l_i = \tilde{l}_i = \frac{k}{2}\,, \qquad m_i = t + \frac{k+2}{2} \mod k+2\,, \qquad s_i = -\tilde{s}_i - 2\,. \tag{3.9}$$

Combination of the 'direct' case (3.8) and the 'flipped' case (3.9) for the two factors $i = 1,\, 2$ yields the four possibilities

1. direct-direct (all values of l_1, l_2, no field identification necessary),
2. direct-flipped ($l_2 = \frac{k}{2}$ with field identification, all l_1),
3. flipped-direct ($l_1 = \frac{k}{2}$ with field identification, all l_2),
4. flipped-flipped ($l_1 = l_2 = \frac{k}{2}$).

Since the automorphism condition is to be matched for both factors $i = 1,\, 2$, the charge projection (3.2) gives a further restriction on the twist sectors. In the four cases, the charge projection is

1. $2t = \frac{k+2}{2} s_1 \mod k+2$. From this, we obtain states in the Neveu-Schwarz sector ($s_i = 0 \mod 2$) if $t = 0 \mod \frac{k+2}{2}$; these direct states have $m_i = t$, and therefore the l_i will take the values $l_i = t \mod 2$. On the other hand, we obtain states in the Ramond sector ($s_i = 1 \mod 2$) if $t = \frac{k+2}{4} \mod \frac{k+2}{2}$; their l labels take the values $l_i = t+1 \mod 2$.

2. $2t = \frac{k+2}{2}(s_1 + 1) \mod k+2$. Here the first factor gives an Ishibashi state by flipping the right-moving representation, and $l_1 = \frac{k}{2}$. The m labels are $m_1 = t + \frac{k+2}{2}$, and $m_2 = t$, since the second factor is unflipped. We will get Ramond states for $t = 0 \mod \frac{k+2}{2}$, and the alignment of the coset labels of the first factor tells us that t has to be even. The label l_2 is odd. Furthermore, we will get NS states for $t = \frac{k+2}{4} \mod \frac{k+2}{2}$ odd, where again l_2 only takes odd values.

3. the same as in case 2, and we obtain the same result as there with interchanged indices $(1 \leftrightarrow 2)$.

4. $2t = \frac{k+2}{2} s_1 \mod k+2$, which is the same as in case 1, but this time both factors have flipped right-moving labels. Therefore, $l_i = \frac{k}{2}$, and $m_i = t + \frac{k+2}{2}$ $(i = 1,\, 2)$. This

gives additional Ishibashi states in the Ramond sector for $t = \frac{k+2}{4} \bmod \frac{k+2}{2}$ even, and contributes an additional state in the Neveu-Schwarz sector if $t = \frac{k+2}{2}$ is odd.

The labels of representations in which Ishibashi states appear are listed in table 1 for the different values of k. With the $k = 2$ model in mind, we will now focus on the case where $\frac{k+2}{4}$ is odd, i.e. $k = 2 \bmod 8$. According to table 1, the Ishibashi states in the twist sector labelled by $t = \nu\frac{k+2}{4}$ form three groups, one where the l labels are both even and the charge labels $m_i = \nu\frac{k+2}{4} = t$ show that we are dealing with an 'unflipped' case in both factors, and two groups where one of the l labels takes the value $\frac{k}{2}$ and the corresponding charge label is shifted to $m = \frac{k+2}{4}(\nu + 2)$, thus indicating a 'flipped' case, while the other factor has l odd and is unflipped. Note that there are no states where it was necessary for both factors to switch the field labels.

The standard tensor product branes at $k = 2 \bmod 8$ are given by

$$\|L_1,\, M_1,\, S_1;\, L_2,\, M_2,\, S_2\rangle\rangle \,=\, (k+2) \sum_{\substack{\nu\in\mathbb{Z}_4,\\ s_1,s_2}} \sum_{\substack{l_1,l_2\\ \text{even}}} \frac{S_{L_1,M_1,S_1;l_1,\nu\frac{k+2}{4},s_1} S_{L_2,M_2,S_2;l_2,\nu\frac{k+2}{4},s_2}}{\sqrt{S_{0,0,0;l_1,\nu\frac{k+2}{4},s_1} S_{0,0,0;l_2,\nu\frac{k+2}{4},s_2}}} \times |l_1, \nu\frac{k+2}{4}, s_1; l_2, \nu\frac{k+2}{4}, s_2\rangle\rangle\,, \tag{3.10}$$

where the s_i obey $l_i + m_i + s_i$ even for $i = 1, 2$. Note that these branes only couple to Ishibashi states in representations with even l labels; no flipped states are involved so far. The formula (3.10) only makes sense for $L_i + M_i + S_i$ even. We will be interested in an alignment $\eta_1 = \eta_2 = \eta$, and hence restrict ourselves to $S_1 - S_2$ even.

There are the following identifications for the brane labels: First, we have the analogue of the field identification, $(L_i,\, M_i,\, S_i) = (k-L_i,\, M_i+k+2,\, S_i+2)$ for either $i = 1$ or $i = 2$. Second, we have the identification $L_i = k-L_i$, again for either $i = 1$ or $i = 2$. Furthermore, we notice that all branes with the same value of $M_1 + M_2 \bmod 8$ are identical, and since s_1-s_2 is even, we also have $(S_1,\, S_2) = (S_1+2,\, S_2+2)$. Last, a shifting $S_1+S_2 \mapsto S_1+S_2+2$ is equivalent to shifting $M_1 + M_2 \mapsto M_1 + M_2 + 4$.

Given these identifications, we conclude that there are $2k^2$ inequivalent branes of type (3.10), k^2 for S_i odd and k^2 for S_i even. In our case, where $k = 2$, we will hence have 8 of these branes, or 4 if we restrict to both S_1 and S_2 even.

The overlap of two of these branes,

$$\langle\langle \hat{L}_1, \hat{M}_1, \hat{S}_1; \hat{L}_2, \hat{M}_2, \hat{S}_2\| q^{\frac{1}{2}(L_0+\tilde{L}_0)-\frac{c}{12}} \|L_1,\, M_1,\, S_1;\, L_2,\, M_2,\, S_2\rangle\rangle\,,$$

can be expressed in the open string sector by means of the modular S transformation. The tensor product of representations $[l_1', m_1', s_1'] \otimes [l_2', m_2', s_2']$ appears in the open string sector with multiplicity

$$\left(\mathcal{N}^{\hat{L}_1}_{L_1,l_1'} + \mathcal{N}^{\hat{L}_1}_{k-L_1,l_1'}\right) \left(\mathcal{N}^{\hat{L}_2}_{L_2,l_2'} + \mathcal{N}^{\hat{L}_2}_{k-L_2,l_2'}\right) \delta^{(2)}(S_1 - \hat{S}_1 + s_1') \times \delta^{(2)}(S_2 - \hat{S}_2 + s_2')\, \delta^{(4)}\Big(\frac{1}{2}(M_1 - \hat{M}_1 + m_1' + M_2 - \hat{M}_2 + m_2') -(S_1 - \hat{S}_1 + s_1' + S_2 - \hat{S}_2 + s_2')\Big)\,. \tag{3.11}$$

I. $\frac{k+2}{2}$ odd						
t	m_1 mod $k+2$	s_1	l_1	m_2 mod $k+2$	s_2	l_2
0	0	even	even	0	even	even
	0	odd	odd	$\frac{k+2}{2}$	odd	$\frac{k}{2}$
	$\frac{k+2}{2}$	odd	$\frac{k}{2}$	0	odd	odd
$\frac{k+2}{2}$	$\frac{k+2}{2}$	even	odd	$\frac{k+2}{2}$	even	odd
	0	even	$\frac{k}{2}$	0	even	$\frac{k}{2}$
II. $\frac{k+2}{4}$ odd						
t	m_1 mod $k+2$	s_1	l_1	m_2 mod $k+2$	s_2	l_2
0	0	even	even	0	even	even
	0	odd	odd	$\frac{k+2}{2}$	odd	$\frac{k}{2}$
	$\frac{k+2}{2}$	odd	$\frac{k}{2}$	0	odd	odd
$\frac{k+2}{4}$	$\frac{k+2}{4}$	odd	even	$\frac{k+2}{4}$	odd	even
	$\frac{k+2}{4}$	even	odd	$3\frac{k+2}{4}$	even	$\frac{k}{2}$
	$3\frac{k+2}{4}$	even	$\frac{k}{2}$	$\frac{k+2}{4}$	even	odd
$\frac{k+2}{2}$	$\frac{k+2}{2}$	even	even	$\frac{k+2}{2}$	even	even
	0	odd	$\frac{k}{2}$	$\frac{k+2}{2}$	odd	odd
	$\frac{k+2}{2}$	odd	odd	0	odd	$\frac{k}{2}$
$3\frac{k+2}{4}$	$3\frac{k+2}{4}$	odd	even	$3\frac{k+2}{4}$	odd	even
	$\frac{k+2}{4}$	even	$\frac{k}{2}$	$3\frac{k+2}{4}$	even	odd
	$3\frac{k+2}{4}$	even	odd	$\frac{k+2}{4}$	even	$\frac{k}{2}$
III. $\frac{k+2}{4}$ even						
t	m_1 mod $k+2$	s_1	l_1	m_2 mod $k+2$	s_2	l_2
0	0	even	even	0	even	even
	0	odd	odd	$\frac{k}{2}$	odd	$\frac{k}{2}$
	$\frac{k+2}{2}$	odd	$\frac{k}{2}$	0	odd	odd
$\frac{k+2}{4}$	$\frac{k+2}{4}$	odd	odd	$\frac{k+2}{4}$	odd	odd
	$3\frac{k+2}{4}$	odd	$\frac{k}{2}$	$3\frac{k+2}{4}$	odd	$\frac{k}{2}$
$\frac{k+2}{2}$	$\frac{k+2}{2}$	even	even	$\frac{k+2}{2}$	even	even
	0	odd	$\frac{k}{2}$	$\frac{k+2}{2}$	odd	odd
	$\frac{k+2}{2}$	odd	odd	0	odd	$\frac{k}{2}$
$3\frac{k+2}{4}$	$3\frac{k+2}{4}$	odd	odd	$3\frac{k+2}{4}$	odd	odd
	$\frac{k+2}{4}$	odd	$\frac{k}{2}$	$\frac{k+2}{4}$	odd	$\frac{k}{2}$

Table 1: The possible Ishibashi states for different parities of k.

We can see from this formula that the open string vacuum appears with multiplicity 2 if either L_1 or L_2 is equal to $\frac{k}{2}$, and with multiplicity 4 if both $L_1 = L_2 = \frac{k}{2}$. The branes of the first kind, where the open string vacuum is contained twice, must be resolved, which yields for $L_1 = \frac{k}{2}$, $L_2 \neq \frac{k}{2}$

$$\begin{aligned}\|\frac{k}{2},\, M_1,\, S_1;\, L_2,\, M_2,\, S_2\rangle\rangle = &\frac{1}{2}\|\frac{k}{2},\, M_1,\, S_1;\, L_2,\, M_2,\, S_2\rangle\rangle_{\text{unresolved}}\\ &+\frac{k+2}{\sqrt{8}}\sum_{\substack{\nu\in\mathbb{Z}_4,l \text{ odd}\\ s_1,s_2}}\sum e^{i\frac{\pi}{4}M_1(\nu+2)-i\frac{\pi}{2}S_1s_1}\frac{S_{L_2,M_2,S_2;l,\nu\frac{k+2}{4},s_2}}{\sqrt{S_{0,0,0;l,\nu\frac{k+2}{4},s_2}}}\\ &\times|\frac{k}{2},\frac{k+2}{4}(\nu+2),s_1;l,\nu\frac{k+2}{4},s_2\rangle\rangle\,,\end{aligned} \tag{3.12}$$

where $\|\frac{k}{2},\, M_1,\, S_1;\, L_2,\, M_2,\, S_2\rangle\rangle_{\text{unresolved}}$ stands for a boundary state of the form (3.10). Note that we can drop the usual factor ± 1 in front of the additional part, since this would only give us another identification in the set of the boundary state labels, namely $(L_2, \pm 1) \equiv (k - L_2, \mp 1)$. The same values of $M_1 + M_2$ do not necessarily encode the same brane any longer; however, shifting $L_2 \mapsto k - L_2$ is compensated by $M_1 \mapsto M_1 + 2$. Altogether, we find $8k$ different states with $S_1 - S_2$ even of this type; $4k$ branes with both S_i even, and $4k$ branes with S_i odd. For $k = 2$, we then have 8 branes with S_i even.

For $L_1 \neq \frac{k}{2}$, $L_2 = \frac{k}{2}$, we have the analogous formula

$$\begin{aligned}\|L_1,\, M_1,\, S_1;\, \frac{k}{2},\, M_2,\, S_2\rangle\rangle = &\frac{1}{2}\|L_1,\, M_1,\, S_1;\, \frac{k}{2},\, M_2,\, S_2\rangle\rangle_{\text{unresolved}}\\ &+\frac{k+2}{\sqrt{8}}\sum_{\substack{\nu\in\mathbb{Z}_4,l \text{ odd}\\ s_1,s_2}}\sum \frac{S_{L_1,M_1,S_1;l,\nu\frac{k+2}{4},s_1}}{\sqrt{S_{0,0,0;l,\nu\frac{k+2}{4},s_1}}}e^{i\frac{\pi}{4}M_2(\nu+2)-i\frac{\pi}{2}S_2s_2}\\ &\times|l,\frac{k+2}{4}\nu,s_1;\frac{k}{2},\frac{k+2}{4}(\nu+2),s_2\rangle\rangle\,,\end{aligned} \tag{3.13}$$

for again $8k$ different states. The branes (3.12), (3.13) thus get resolved by means of the flipped Ishibashi states; branes with $L_1 = \frac{k}{2}$, $L_2 \neq \frac{k}{2}$ couple to representations where we need the field identification in the first factor, and branes with $L_1 \neq \frac{k}{2}$, $L_2 = \frac{k}{2}$ couple to representations that are flipped in the second factor. Since there are no states in which we had to use the field identification in both factors, it seems reasonable that the branes at $L_1 = L_2 = \frac{k}{2}$ do not couple to any flipped Ishibashi state. This is indeed the case, and we find the formula

$$\|\frac{k}{2},M_1,S_1;\frac{k}{2},M_2,S_2\rangle\rangle = \frac{1}{2}\,\|\frac{k}{2},M_1,S_1;\frac{k}{2},M_2,S_2\rangle\rangle_{\text{unresolved}} \tag{3.14}$$

for these branes. There are 8 different branes of this kind, 4 branes with S_i even and 4 with S_i odd.

3.1.2 Permutation boundary states

Permutation boundary states of type B satisfy the gluing conditions

$$\begin{aligned}(L_n^{(1)} - \tilde{L}_{-n}^{(2)})\|B\rangle\rangle = (L_n^{(2)} - \tilde{L}_{-n}^{(1)})\|B\rangle\rangle &= 0\,,\\ (J_n^{(1)} + \tilde{J}_{-n}^{(2)})\|B\rangle\rangle = (J_n^{(2)} + \tilde{J}_{-n}^{(1)})\|B\rangle\rangle &= 0\,,\\ (G_r^{\pm\,(1)} + i\eta\tilde{G}_{-r}^{\pm\,(2)})\|B\rangle\rangle = (G_r^{\pm\,(2)} + i\eta\tilde{G}_{-r}^{\pm\,(1)})\|B\rangle\rangle &= 0\,.\end{aligned}$$

Whenever we have to distinguish explicitly between tensor product and permutation boundary states we will denote the latter with an additional superscript σ, $\|B\rangle\rangle^{\sigma}$. The permutation boundary states have been worked out in [16, 17, 19]:

$$\|L, M, \hat{M}, S_1, S_2\rangle\rangle = \frac{1}{k+2} \sum_{\nu\in\mathbb{Z}_4} \sum_{l,m} \sum_{s_1,s_2} \frac{S_{Ll}}{S_{0l}} e^{i\frac{\pi}{4}\hat{M}\nu+i\frac{\pi}{k+2}Mm-i\frac{\pi}{2}(S_1s_1+S_2s_2)} \times|l, m+n, s_1; l, -m+n, s_2\rangle\rangle\,, \tag{3.15}$$

where the sums over l and m run over appropriate values in the twist sector $n = \nu\frac{k+2}{4}$, and $s_i = l+m+n \mod 2$. In this formula, we have the label constraints $L+M+S_1+S_2 = 0 \mod 2$ and $M-\hat{M} = 0 \mod 2$, and we restrict ourselves again to states with $S_1-S_2 = 0 \mod 2$.

For $L \neq \frac{k}{2}$, there is again an analogue of the field identification, $(L, M, \hat{M}, S_1+S_2) = (k-L, M+k+2, \hat{M}+4, S_1+S_2+2)$. From the charge projection (3.2) we see that $(\hat{M}, S_1+S_2) = (\hat{M}+4, S_1+S_2+2)$, which we can combine with the field identification to yield $(L, M) = (k-L, M+k+2)$. Furthermore, states with (S_1, S_2) and (S_1+2, S_2+2) are again the same. We conclude that there are $4k(k+2)$ different permutation branes with $L \neq \frac{k}{2}$ and S_1-S_2 even. If $k=2$ there are thus 16 different branes with $L \neq 1$ and even S_i.

For $L = \frac{k}{2}$, $(L, M) = (k-L, M+k+2)$ is an identification on its own, without making use of $(\hat{M}, S_1+S_2) \mapsto (\hat{M}+4, S_1+S_2+2)$. There are hence $4(k+2)$ different permutation branes with $L = \frac{k}{2}$ and S_1-S_2 even; for $k=2$ this leaves us with 8 different $L=1$ branes at even S_i.

Altogether, there are $4k^2+12k+8$ different permutation branes with S_1-S_2 even for $k = 2 \mod 8$, compared to a total of $2k^2+16k+8$ tensor product branes.

4. Comparison of torus and Gepner model

We will now focus on the case where $k=2$, and compare the boundary states we have just described with those of the torus from section 2.3.

In order to compare the two respective classes of Ishibashi states we will write the Gepner model Ishibashi states in terms of Ishibashi states of the diagonal $N=2$ algebra. An Ishibashi state in the left-moving representation

$$(h_1, q_1)\otimes(h_2, q_2) = \bigoplus_{[(H,Q)]} (H, Q)\,, \tag{4.1}$$

where the direct sum runs over the diagonal representations of highest weight H and charge Q (see (3.4)), consists of a sum of 'diagonal' Ishibashi states up to the choice of phases $\psi_{(H,Q)}$,

$$|h_1, q_1, h_2, q_2\rangle\rangle = \sum_{[(H,Q)]} e^{i\psi_{(H,Q)}}|H, Q\rangle\rangle_{(h_1,q_1)\otimes(h_2,q_2)}\,. \tag{4.2}$$

In general we can not always set these phases to zero, since there exist tensor products of minimal model representations that admit both a tensor product and a permutation

Ishibashi state, and for those the phases $e^{i\psi_{(H,Q)}}$ have to be different. This is the case whenever $h_1 = h_2$ and $q_1 = q_2$ in (4.1). Let us define the diagonal Ishibashi states in (4.2) for $h_1 = h_2$ and $q_1 = q_2$ such that all phases $\psi_{(H,Q)}$ vanish for the permutation Ishibashi state. Then, as explained in [19], the phase $\psi_{(2h_1,2q_1)}$ of the ground state in the tensor product Ishibashi state is

$$\psi_{(2h_1,2q_1)} = \frac{\pi}{2}s - \frac{\pi}{k+2}m\,, \tag{4.3}$$

where $(l,\, m,\, s)$ are the coset labels of the representation $(h_1,\, q_1)$.

4.1 Brane dictionary from the $\mathbb{Z}_4$ symmetries

The identification of the Ishibashi states maps diagonal $N = 2$ Ishibashi states of the Gepner model to $N = 2$ Ishibashi states of the torus with the same highest weight and charge. In the vacuum sector, this is already sufficient to identify the Ishibashi states. Consider for example the diagonal Ishibashi state at $H = \frac{1}{2}$, $Q = 1$, which appears on the Gepner side only in the left-moving representation $h_1 = h_2 = \frac{1}{4}$, $q_1 = q_2 = \frac{1}{2}$ in the twist sector $t = 2$. Since this is the only Ishibashi state with these quantum numbers, we can identify it with the state $|\frac{1}{2},\, 1\rangle\rangle$ on the torus.[2] On the Gepner side, this Ishibashi state couples to a permutation brane $(L,\, M,\, \hat{M},\, S_1,\, S_2)$ with the factor

$$\frac{1}{\sqrt{2}} \sin\left(\frac{\pi}{4}(L+1)\right) e^{i\frac{\pi}{2}\hat{M}}\,,$$

while on the torus side the coupling is $\mathcal{N}(\phi)e^{i\pi\phi}$. Hence we deduce that the angle of the permutation brane is

$$\phi = \frac{\pi}{2}\hat{M}\,. \tag{4.4}$$

On the Gepner side, we also find a tensor product Ishibashi state in the same representation, whose coefficient will analogously yield the angle ϕ in terms of the tensor product brane labels. Remembering the additional phase (4.3), we find with the coefficients from the formulae (3.10) – (3.14)

$$\phi = \frac{\pi}{2}(M_1 + M_2 + 1) \tag{4.5}$$

for a tensor product brane $(L_1,\, M_1,\, S_1,\, L_2,\, M_2,\, S_2)$.

In the more general cases of vanishing diagonal charge ($Q = 0$), the identification of the diagonal Gepner Ishibashi states at highest weight $H > 0$ with the torus Ishibashi states is more complicated. However we can use that eigenstates of the two $\mathbb{Z}_4$ symmetries on the Gepner side will be mapped to eigenstates of the corresponding symmetries on the torus side.

As an example, consider the identification of the diagonal Ishibashi states of lowest nonvanishing highest weight in the NS sector, which are the states $H = \frac{1}{4}$, $Q = 0$. In the Gepner model, these states appear in the left-moving representations $(\frac{1}{8},\, \pm\frac{1}{4}) \otimes (\frac{1}{8},\, \mp\frac{1}{4})$,

[2] Strictly speaking this only fixes the identification up to a phase. As we shall see, it is consistent that we choose this phase factor to be trivial.

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whose Ishibashi states are of permutation type in the twist sectors $n = 0$ and $n = 2$ and of tensor product type in the other sectors. Let us choose the basis on the torus to be

$$|1,\, 0,\, \pi;\, \eta\rangle\rangle^{t,1}, \qquad |1,\, 0,\, \pi;\, \eta\rangle\rangle^{t,3} \qquad (t = 0, \ldots, 3) \tag{4.6}$$

in the notation (2.16). The ansatz for the identification is then

$$\begin{aligned} |\left(\frac{1}{8}, -\frac{1}{4}\right) \otimes \left(\frac{1}{8}, \frac{1}{4}\right)\rangle\rangle^t \quad &\leftrightarrow \quad \alpha^{(t)}|1,\, 0,\, \pi;\, \eta\rangle\rangle^{t,1}\,, \\ |\left(\frac{1}{8}, \frac{1}{4}\right) \otimes \left(\frac{1}{8}, -\frac{1}{4}\right)\rangle\rangle^t \quad &\leftrightarrow \quad \beta^{(t)}|1,\, 0,\, \pi;\, \eta\rangle\rangle^{t,3} \end{aligned}$$

for $0 \leq t \leq 3$, where the phases $\alpha^{(t)}$ and $\beta^{(t)}$ are initially undetermined. For a permutation brane $(L,\, M,\, \hat{M},\, S_1,\, S_2)$ we then obtain

$$\begin{aligned} e^{iA} &= e^{-iA} = \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}M + i\frac{\pi}{2}L - i\pi S_2}\left(\alpha^{(0)} + \beta^{(0)} e^{i\frac{\pi}{2}M}\right), \\ e^{iB} &= e^{-iB} = \frac{i}{\sqrt{2}} e^{i\frac{\pi}{4}M + i\frac{\pi}{2}L - i\pi S_2}\left(\alpha^{(0)} - \beta^{(0)} e^{i\frac{\pi}{2}M}\right), \\ \alpha^{(2)} &= -\beta^{(0)}\,, \\ \beta^{(2)} &= -\alpha^{(0)}\,. \end{aligned} \tag{4.7}$$

Here, A and B are position and Wilson line of the torus brane. The label L is even, since these are the only permutation branes that couple to the considered Ishibashi states. A solution to the equations (4.7), i.e. a consistent formula for position A and Wilson line B in terms of the permutation brane labels, can be given for $\alpha^{(0)} = (\beta^{(0)})^* = -(\alpha^{(2)})^* = -\beta^{(2)} = e^{i\frac{\pi}{4}}$ (see table 2).[3] We find similar consistency equations for A and B from the tensor product branes at a single fixed point (3.12), (3.13).

Positions and Wilson lines of the other tensor product and permutation branes (at $L = 1$ or $L_1 + L_2$ even, respectively) can be obtained from matching the states at $H = \frac{1}{2}$, $Q = 0$ in a similar way as in the case $H = \frac{1}{4}$, $Q = 0$ we have just mentioned. The formulae for positions and Wilson lines were also checked in the R sector, and for higher values of H.

The procedure provides a consistent map for the positions and Wilson lines of the images of tensor product and permutation branes on the torus, which — for a certain phase chioce — is given in table 2. From this table, we see that the branes coupling to the Ishibashi states at lowest momentum (highest weight $H = \frac{1}{4}$) have L even (permutation branes) or $L_1 + L_2$ odd (resolved tensor product branes coupling to flipped states). These branes are the 'short' or 'light' branes, i.e. those that couple to the vacuum with the lowest coefficient; their angles are integer multiples of π. For the permutation branes at L even, shifting the M label by 2 leads to a relative phase shift between position and Wilson line. In the case of the tensor product branes at $L_1 + L_2$ odd, the relative phase between position and Wilson line is changed by passing from the branes with $L_1 = 1$ to those with $L_2 = 1$ (and vice versa).

[3] Again, there exist other possible phase choices. These correspond to choosing the absolute position and orientation of one reference brane.

Permutation branes	
$\phi = \frac{\pi}{2}\hat{M}$	
$L = 0 \bmod 2$	$M = 0 \bmod 4\ :\quad A = \frac{\pi}{2}L + \frac{\pi}{4}M + \pi S_1\,,\qquad B = A$ $M = 2 \bmod 4\ :\ A = \frac{\pi}{2}L + \frac{\pi}{4}(M-2) + \pi S_1\,,\ B = A + \pi$
$L = 1$	$A = \frac{\pi}{2}(M-1)\,,\ B = A + \pi$
Tensor product branes	
$\phi = \frac{\pi}{2}(M_1 + M_2 + 1)$	
$L_1 = 1,\ L_2 = 0 \bmod 2$	$A = \frac{\pi}{2}L_2 + \frac{\pi}{2}M_1 + \pi S_1\,,\ B = A$
$L_1 = 0 \bmod 2,\ L_2 = 1$	$A = \frac{\pi}{2}L_1 + \frac{\pi}{2}M_2 + \pi(S_1 + 1)\,,\ B = A + \pi$
$L_1,\ L_2 = 0 \bmod 2$	$A = B = 0$
$L_1 = L_2 = 1$	$A = B = \pi$

Table 2: Example for a consistent choice of positions A and Wilson lines B in terms of the coset labels for the images of the B-type permutation and tensor product branes, with $\epsilon = e^{-i\frac{\pi}{2}(S_1+S_2)}$.

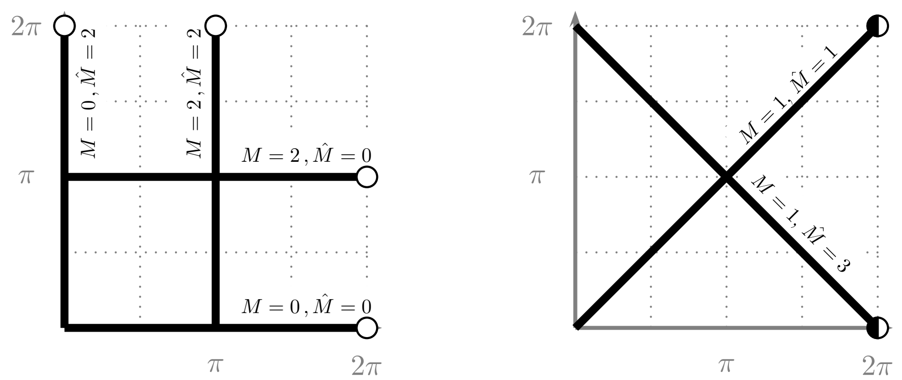

Table 3: Permutation brane positions in the labels of table 2. The left diagram shows the short branes ($L = 0$) at different values of M and $\hat{M}$, the right diagram contains examples of the long branes ($L = 1$). The fermion structure has been set to $\eta = +1$, $S_1 = S_2 = 0$. The filling of the circles at the end of the lines denotes the Wilson line of the brane; empty circles correspond to Wilson line $B = 0$, half-filled circles to Wilson line $B = \pi$.

5. Relation to matrix factorisations

Our simple model corresponds to an orbifold of the Landau-Ginzburg superpotential

$$W = x_1^4 + x_2^4 + z^2\,, \tag{5.1}$$

where the presence of the trivial factor z^2 is related to the charge projection of our Gepner model. Topological branes are given by a pair

$$\left(Q = \begin{pmatrix} 0 & J \\ E & 0 \end{pmatrix}\,,\ \gamma\right) \tag{5.2}$$

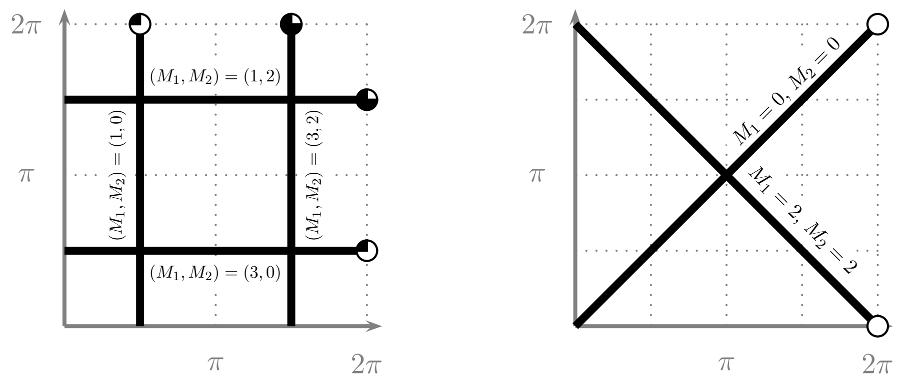

Table 4: Tensor product brane positions in the labels of table 2. The left diagram shows the resolved short tensor product branes with $L_1 = 1$, $L_2 = 0$, the right diagram contains long branes with $L_1 = L_2 = 0$. The fermion structure has been set to $\eta = +1$, $S_1 = S_2 = 0$. The filling of the circles at the end of the lines denotes the Wilson line of the brane; empty circles correspond to Wilson line $B = 0$, quarter-filled circles to Wilson line $B = \frac{\pi}{2}$, etc.

where Q has entries that are polynomials in x_1, x_2, and z such that it factorises the superpotential, i.e.

$$Q^2 = W\mathbf{1}\,. \tag{5.3}$$

The orbifold matrix γ satisfies

$$\gamma Q(ix_1, ix_2, -z)\gamma^{-1} = Q(x_1, x_2, z) \qquad \text{and} \qquad \gamma^4 = \mathbf{1}\,. \tag{5.4}$$

The orbifold matrix is hence only defined up to a phase factor $e^{i\frac{\pi}{2}n}$ for $n = 0,\, 1,\, 2,\, 3$.

The relative couplings of factorisations to the RR-primary ground states can be computed from a general formula given in [35] (for a review, see e.g. [36]). In our case, these states are in the following left-moving representations:

$$\begin{array}{lll} t & (l_1,\, m_1,\, s_1) \otimes (l_2,\, m_2,\, s_2) & (h_1,\, q_1) \otimes (h_2,\, q_2) \\ 1 & \quad (0,\, 1,\, 1) \otimes (0,\, 1,\, 1) & \quad (\frac{1}{16}, \frac{1}{4}) \otimes (\frac{1}{16}, \frac{1}{4}) \\ 3 & \quad (0,\, 7,\, 3) \otimes (0,\, 7,\, 3) & (\frac{1}{16}, -\frac{1}{4}) \otimes (\frac{1}{16}, -\frac{1}{4}) \end{array} \tag{5.5}$$

As above, t denotes the twist sector. The R primary field with $(h,\, q) = (\frac{1}{16},\, 0)$ appears only in combination with the field $(h,\, q) = (\frac{5}{16},\, \frac{1}{2})$, which is not primary. Since the RR primary states all appear in twisted sectors, the formula for their brane couplings reduces to

$$C(Q,\, \gamma;\, t) = \mathrm{Str}(\gamma^t) \qquad (t = 1,\, 3)\,, \tag{5.6}$$

Str denoting the supertrace. In the following we are going to identify the Gepner branes of section 3.1 with certain matrix factorisations by computing the couplings of different factorisations and comparing the results to the couplings obtained from the brane formulae. From now on we will set $S_1 = S_2 = 0$.

Let us first consider the matrix factorisations corresponding to the permutation branes (3.15), which have been worked out in general in [19]. The analogues of the rank 1 factorisations described in [19] are given by matrices of the form

$$\begin{aligned} J &= \begin{pmatrix} \prod_{\eta\in\mathcal{I}}(x_1-\eta x_2) & -z \\ z & \prod_{\eta\in\mathcal{I}^C}(x_1-\eta x_2) \end{pmatrix}, \\ E &= \begin{pmatrix} \prod_{\eta\in\mathcal{I}^C}(x_1-\eta x_2) & z \\ -z & \prod_{\eta\in\mathcal{I}}(x_1-\eta x_2) \end{pmatrix}, \\ \gamma &= \mathrm{diag}(1,-i^{|\mathcal{I}|},i^{|\mathcal{I}|},-1)\times e^{i\frac{\pi}{2}n}, \end{aligned} \tag{5.7}$$

where $\mathcal{I}$ is a subset of the set of fourth roots of -1, $\mathcal{I}^C$ is its respective complement, and $|\mathcal{I}|$ is the number of elements in $\mathcal{I}$. These factorisations are identified with branes in the Gepner model in the following way:

(i) Factorisations of type (5.7) with $|\mathcal{I}|=1$ or $|\mathcal{I}|=3$ correspond to the permutation branes (3.15) with $L\neq 1$. There are 8 factorisations of this type, and each has 4 values for the phase of γ. A factorisation with $|\mathcal{I}|=3$ and phase $e^{i\frac{\pi}{2}n}$ is identical to a factorisation with $\mathcal{I}^C$ and phase $e^{i\frac{\pi}{2}(n+1)}$, so that we are left with 16 different branes, as we have expected from the counting in 3.1.2. Without loss of generality, we can restrict ourselves to permutation branes with $L=0$. Taking e.g. $\|L=0,\, M=0,\, \hat{M}=0,\, S_i=0\rangle\rangle$ to be the factorisation with $\mathcal{I}=\{e^{i\frac{\pi}{4}}\}$ and $n=0$, we find that the $L=0$ branes correspond to factorisations $|\mathcal{I}|=1$ with $\hat{M}=2n$. The values of $M\in\{0,\,2,\,4,\,6\}$ correspond to the choice of the element in $\mathcal{I}$.

(ii) The factorisations of type (5.7), where $\mathcal{I}$ contains two consecutive roots of -1, correspond to the permutation branes (3.15) with $L=1$. There are two of these factorisations, each with four choices of γ, and they are again pairwise identified in a similar way as before, so that there are 4 different branes. An identification consistent with the one of the $L=0$ permutation branes from above yields $\hat{M}=2n+1$.

(iii) The missing factorisations of type (5.7) correspond to the resolved tensor product branes at $L_1=L_2=1$, as it has already been argued on general grounds in [37]. The missing factorisations are those two for which $\mathcal{I}$ contains two non-consecutive roots, and each factorisation has 4 possibilities for γ. As before there is again an identification between pairs that reduces the number of different branes to 4, and the phase $e^{i\frac{\pi}{2}n}$ of γ is linked to the labels M_i by $M_1+M_2=2n$ in our conventions.

We have hence found corresponding Gepner branes for all factorisations (5.7), in agreement with the proposition in [37].

Let us also give the factorisations corresponding to the other tensor product branes. The branes (3.10) with $L_1=L_2=0$ belong to the usual rank 4 tensor product factorisations

$x_1x_1^3+x_2x_2^3+zz$. There are four factorisations of this type, and each comes again with four choices of γ. There are however only four inequivalent factorisations, which are given by

$$
\begin{aligned}
J &= \begin{pmatrix} x_1 & -x_2^3 & -z & 0 \\ x_2 & x_1^3 & 0 & -z \\ z & 0 & x_1^3 & x_2^3 \\ 0 & z & -x_2 & x_1 \end{pmatrix}, \qquad E = \begin{pmatrix} x_1^3 & x_2^3 & z & 0 \\ -x_2 & x_1 & 0 & z \\ -z & 0 & x_1 & -x_2^3 \\ 0 & -z & x_2 & x_1^3 \end{pmatrix}, \\
\gamma &= \operatorname{diag}(1,1,-i,i,i,-i,-1,-1) \times e^{i\frac{\pi}{2}n}\,.
\end{aligned} \tag{5.8}
$$

In our conventions we then have $M_1 + M_2 = 2n$.

The resolved tensor product branes (3.12) with $L_1 = 1$, $L_2 \neq 1$ correspond to the rank 2 factorisations $(x_1^2 + iz)(x_1^2 - iz) + x_2x_2^3$. There are four factorisations of this type, each with four choices of γ. They are given by

$$
\begin{aligned}
J &= \begin{pmatrix} x_1^2 - iz & -x_2^3 \\ x_2 & x_1^2 + iz \end{pmatrix}, \qquad E = \begin{pmatrix} x_1^2 + iz & x_2^3 \\ -x_2 & x_1^2 - iz \end{pmatrix}, \\
\gamma &= \operatorname{diag}(1,i,-1,-i) \times e^{i\frac{\pi}{2}n}\,,
\end{aligned} \tag{5.9}
$$

and

$$
\begin{aligned}
J &= \begin{pmatrix} x_1^2 + iz & -x_2^3 \\ x_2 & x_1^2 - iz \end{pmatrix}, \qquad E = \begin{pmatrix} x_1^2 - iz & x_2^3 \\ -x_2 & x_1^2 + iz \end{pmatrix}, \\
\gamma &= \operatorname{diag}(1,i,-1,-i) \times e^{i\frac{\pi}{2}n}\,,
\end{aligned} \tag{5.10}
$$

with the two other factorisations arising from (5.9), (5.10) by interchanging $x_2 \leftrightarrow x_2^3$ and $(x_1^2 + iz) \leftrightarrow (x_1^2 - iz)$. However, this interchange leads to equivalent factorisations. We hence find 8 different factorisations, in agreement with the 8 different resolved tensor product branes with $L_1 = 1$, $L_2 \neq 1$. In our conventions, we have $M_1 + M_2 = 2n + 1$.

The other class (3.13) of resolved tensor product branes with $L_1 \neq 1$, $L_2 = 1$ can be identified with the factorisations $x_1x_1^3 + (x_2^2 + iz)(x_2^2 - iz)$, where the different branes are given by (5.9) and (5.10) with $x_1 \leftrightarrow x_2$.

We have thus identified matrix factorisations for all the Gepner branes described in 3.1.

6. Conclusion

In this paper, we have worked out a dictionary between explicit sets of tensor product and permutation branes (3.10), (3.12), (3.13), (3.14), (3.15) in the Gepner construction involving two minimal models at $k = 2$ (3.3), and the branes (2.27) of the torus at the self-dual point. To do this, we have identified the 'natural' $\mathbb{Z}_4$ symmetry (2.12) on the torus with the quantum symmetry in the Gepner model, and the $\mathbb{Z}_4$ symmetry involving a T-duality transformation in both torus directions (2.15) with the phase shift $e^{i\frac{\pi}{2}m_1}$. For a convenient choice of some relative phases, this has yielded an identification of angles, positions, and Wilson lines of the torus branes corresponding to the considered branes in the Gepner model in terms of the labels of the latter (table 2).

The $N = 2$ A-type boundary states on the torus can all be given by U(1) branes, satisfying gluing conditions that involve (twice) the angle of the brane on the torus analogously to the electric flux of U(1) branes in electric fields (2.22). Both $\mathbb{Z}_4$ symmetries rotate the angle by 90 degrees (or $\phi \mapsto \phi + \pi$). With our definitions, the values of position and Wilson line of a brane are kept fixed under the first symmetry, and exchanged under the second. Hence the first symmetry can be seen as a mere rotation of the brane around the point (π, π) in the diagrams, leaving the distance to the origin and the Wilson line fixed, while the second symmetry in general involves a shift in position.

The Gepner branes can be identified with matrix factorisations of a corresponding Landau-Ginzburg orbifold. Although the direct identification of Gepner brane labels with e.g. the phases of the orbifold matrices γ is heavily depending on our conventions, we can for the considered model make the more general remark that a shift by $\frac{\pi}{2}$ in the phase of γ corresponds to a rotation of 90 degrees of the corresponding brane on the torus, or a shift of ϕ by π, respectively.

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A. Conventions for the $N = 2$ minimal models

The $N = 2$ algebra is generated by the modes L_n of the energy-momentum tensor, the modes J_n of the U(1) current, and the modes $G_r^{\pm}$ of the two supercharges, where $n \in \mathbb{Z}$ and $r \in \mathbb{Z}$ for the R sector or $r \in \mathbb{Z} + \frac{1}{2}$ for the NS sector. They obey the (anti-)commutation relations

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}\,, \\
[L_m, J_n] &= -nJ_{m+n}\,, \\
[L_m, G_r^{\pm}] &= \left(\frac{m}{2} - r\right) G_{m+r}^{\pm}\,, \\
[J_m, J_n] &= \frac{c}{3}m\delta_{m,-n}\,, \\
[J_m, G_r^{\pm}] &= \pm G_{m+r}^{\pm}\,, \\
\{G_r^+, G_s^-\} &= 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s}\,;
\end{aligned}
$$

all the other (anti-)commutators vanish. The $N = 2$ minimal models at level $k \in \mathbb{N}$ have central charge

$$c = \frac{3k}{k+2} \tag{A.1}$$

and are described by means of the coset construction

$$\frac{su(2)_k \otimes u(1)_4}{u(1)_{2(k+2)}} . \tag{A.2}$$

Highest weight representations of the coset construction are labelled by

$$l \in \{0, \ldots, k\} , \quad m \in \mathbb{Z}_{2(k+2)} , \quad s \in \mathbb{Z}_4 , \tag{A.3}$$

with the selection rule that $l + m + s$ must be even. The corresponding highest weight state is denoted as

$$\Phi^{l,s}_m \equiv |l, m, s\rangle , \tag{A.4}$$

with highest weight and charge

$$h = \frac{l(l+2) - m^2}{4(k+2)} + \frac{s^2}{8} \bmod 1 , \qquad q = \frac{s}{2} - \frac{m}{k+2} \bmod 2 . \tag{A.5}$$

The set of labels $(k - l,\, m + k + 2,\, s + 2)$ gives a representation identical to the one with labels $(l,\, m,\, s)$; we denote the equivalence class by $[l,\, m,\, s]$. A complete $N = 2$ NS representation is given by the direct sum $(l,\, m,\, 0) \oplus (l,\, m,\, 2)$, a complete R representation is $(l,\, m,\, 1) \oplus (l,\, m,\, 3)$, where one part of the direct sum contains the states at even and at odd fermion number respectively. The chiral primaries are given by the labels $(l,\, l,\, 0)$ or $(l,\, -l-2,\, 2)$ in the NS sector and by $(l,\, l+1,\, 1)$ or $(l,\, -l-1,\, -1)$ in the R sector. The modular S matrix of the coset theory is

$$S_{LMS,lms} = \frac{1}{\sqrt{2(k+2)}} S_{Ll} e^{i\frac{\pi}{k+2}Mm} e^{-i\frac{\pi}{2}Ss} , \tag{A.6}$$

where

$$S_{Ll} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}(L+1)(l+1)\right) \tag{A.7}$$

is the modular S matrix of $su(2)_k$. The spectral flow of unit $\frac{1}{2}$ acts on the coset labels by fusion with $(0,\, 1,\, 1)$.

B. Table for the Gepner model and its relation to the torus

A list of the minimal model representations of the $N = 2$ superconformal algebra at $k = 2$ in terms of the coset labels is given in table 5. If one follows the comparison of the characters that lead (3.4) to higher orders, one is lead to the following general formula for diagonal representations at $c = 3$ contained in the tensor product of two minimal models at $k = 2$:

$$\begin{aligned}
(0,\, 0) \otimes (0,\, 0) = (0,\, 0) &\bigoplus_{m\in\mathbb{Z}} \left(\frac{8|m|-1}{2},\, \mathrm{sign}\left(\frac{8m-1}{2}\right)\right) \bigoplus_{n\in\mathbb{N}} (n^2,\, 0) \\
&\bigoplus_{p,q\in\mathbb{N}} (p^2+q^2,\, 0) , \\
\left(\frac{1}{2},\, 0\right) \otimes (0,\, 0) = &\bigoplus_{n\,\mathrm{odd}} \left(\frac{n^2}{2},\, 0\right) \bigoplus_{p^2+q^2\,\mathrm{odd}} \left(\frac{p^2+q^2}{2},\, 0\right) , \\
\left(\frac{1}{8},\, \frac{1}{4}\right) \otimes \left(\frac{1}{8},\, -\frac{1}{4}\right) = &\bigoplus_{n\,\mathrm{odd}} \left(\frac{n^2}{4},\, 0\right) \bigoplus_{p^2+q^2\,\mathrm{odd}} \left(\frac{p^2+q^2}{4},\, 0\right) .
\end{aligned}$$

NS – sector					R – sector				
l	m	s	h	q	l	m	s	h	q
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{\frac{1}{16}}$	$\mathbf{\frac{1}{4}}$
0	2	2	$\frac{1}{4}$	$\frac{1}{2}$	0	3	3	$\frac{9}{16}$	$\frac{3}{4}$
0	4	2	$\frac{1}{2}$	0	0	5	3	$\frac{9}{16}$	$\frac{1}{4}$
$\mathbf{0}$	$\mathbf{6}$	$\mathbf{2}$	$\mathbf{\frac{1}{4}}$	$-\mathbf{\frac{1}{2}}$	$\mathbf{0}$	$\mathbf{7}$	$\mathbf{3}$	$\mathbf{\frac{1}{16}}$	$-\mathbf{\frac{1}{4}}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{\frac{1}{8}}$	$-\mathbf{\frac{1}{4}}$	1	0	1	$\frac{5}{16}$	$\frac{1}{2}$
1	3	2	$\frac{1}{8}$	$\frac{1}{4}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{\frac{1}{16}}$	$\mathbf{0}$

Table 5: List of coset labels, highest weights and charges for the representations of the $N = 2$ algebra at level $k = 2$. For the NS sector, the coset label s indicates the bosonic subalgebra of even fermion number, in the R sector the coset labels give the subalgebra of the highest weight state which is annihilated by G_0^+. Bold face indicates chiral primaries.

Here, m runs over all integers, whereas n, p, and q only take values in the set of natural numbers. By applying the spectral flow on both factors on the left hand side as well as on the summands on the right-hand side, one obtains the formulae for the other tensor products appearing in the Gepner model. This formula has not been proved, but it has been checked numerically up to level 50.

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